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EXACT MAXIMUM SLOPES FOR TRANSIENT MATRIX HEAT-TRANSFER TESTING

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Abstract—Analytical expressions are derived for both the maximum slope of exit fluid temperatures (fluid temperature transient response) due to a step function inlet fluid temperature change, as based on the solution of Hausen's mathematical model, and for the reduced time at which the maximum slope is attained. These expressions are used to demonstrate the existence of a critical N_{tu}-value below which the exit fluid temperatures have no point of inflection. Values, accurate to four significant figures, for both the maximum slope and reduced time are presented up to N_{tu}-values of ten. It is pointed out that in the neighborhood of $N_{tu} = 2$ even very small errors in the determination of the maximum slope can result in significant errors of the corresponding N_{tu}-values. Finally, an asymptotic expression for the maximum slope is presented which is of use for large values of N_{tu}.

NOMENCLATURE

- h, heat-transfer coefficient [Btu/(h ft^2 deg F)];
- A, total heat-transfer area [ft^2];
- W_s, mass of solid in core [lb];
- W_f, mass of fluid [lb];
- w_f, mass-flow rate of fluid [lb/h];
- c_s, specific heat of solid [Btu/(lb degF)];
- c_f, specific heat of fluid [Btu/(lb degF)];
- x, distance from test section inlet [ft];
- L, length of solid core [ft];
- t, time [h];
- G, temperature of the fluid (gas) [°F];
- S, temperature of the solid [°F];
- N_{tu} dimensionless parameter (number of transfer units), $N_{tu} = hA/w_f c_f$.

INTRODUCTION

THE DESIGN of matrix-type gas turbine regenerators requires accurate heat-transfer data. A method of testing the various matrix surfaces currently being used is the so-called transient technique which involves both an experimental apparatus and a mathematical model for the single blow problem. For detailed background information the reader is referred to such standard works as Jakob [1] or Kays and London [2]. In the experimental procedure a sample of the matrix is subjected to some change of the transfusing fluid at the test section inlet and the transient fluid temperature at the test-section exit is recorded.

The mathematical model is provided by Hausen's [3] partial differential equation system. In this model the effects of longitudinal heat conduction will be neglected. The heat-transfer properties of the matrix surface may, in principle, be evaluated by matching the recorded transient exit temperatures with the computed response curves which have been obtained as solutions of Hausen's equation system.

Locke [4] has demonstrated that the matchings of response curves could be avoided by the use of a unique relationship between the maximum slopes of response curves and the number of transfer units, N_{tu}, for the case that the inlet gas temperature change is a step function. Locke's method, known as "maximum slope technique" has proved to be a valuable tool. The numerical data which were first presented by Locke [4] were based on an approximation which is questionable for very low values of N_{tu}; Locke's data have subsequently been corrected by improved finite difference solutions of Hausen's equation system and appear as such in Kays and London [2].

It is the purpose of this paper to show that although the relationship between the maximum slope and N_{tu} is unique for all values of N_{tu} this relationship is nevertheless singular in its behavior for $N_{tu} = 2$. The object of this paper is to point out a limitation of an accepted method and to issue a warning against its unrestricted use; it does not offer a new or an improved method. It is obvious that a convincing case should be based on exact analysis. In this paper the dependence of the maximum slope on N_{tu} will be expressed by analytic expressions and data will be presented which are based on exact analysis. It will be shown that the exit fluid temperatures have for $N_{tu} < 2$ no points of inflection. It will also be shown that the exit fluid temperatures (transient response curves) have for $N_{tu} < 2$ their maximum slopes in the instant when the temperature front of the fluid reaches the test-section exit. The fact will be pointed out that in the neighborhood of $N_{tu} = 2$ even very small errors in the determination of the maximum slope can result in significant errors of the corresponding N_{tu}-values.

PROBLEM FORMULATION AND ANALYTIC SOLUTION

The heat transfer from a fluid transfusing through a porous solid core to the latter can be described by the following system of partial differential equations:

$$\frac{\partial G}{\partial t} + \frac{w_f L}{W_f}\frac{\partial G}{\partial x} = \frac{h A}{c_f W_f}(S - G), \tag{1}$$

$$\frac{\partial S}{\partial t} = \frac{h A}{c_s W_s}(G - S). \tag{2}$$

Equations equivalent to (1), (2) were stated by Hausen [3] and, independently, by Schumann [5]. For a discussion of the assumptions which lead to these equations, and for further reference, see Jakob [1].

It is practical to introduce a generalized position variable

$$z = \frac{h A}{w_f c_f}\cdot\frac{x}{L} \tag{3}$$

and a generalized time variable

$$\tau = \frac{h A}{W_s c_s}\left(t - \frac{W_f}{w_f}\frac{x}{L}\right). \tag{4}$$

Consequently, in terms of the new independent variables z and τ, equations (1) and (2) become

$$\frac{\partial G}{\partial z} + G = S, \tag{5}$$

$$\frac{\partial S}{\partial \tau} + S = G. \tag{6}$$

For z and τ the designations Hausen's reduced variables, or Nusselt's variables, are sometimes used.

The analytic solution of the characteristic initial value problem, or Goursat problem, as posed by equations (5) and (6) together with the boundary conditions expressed by

$$G(z,\tau)\big|_{z=0} = G(0,\tau) \equiv g(\tau), \tag{7}$$

$$S(z,\tau)\big|_{\tau=0} = S(z,0) \equiv s(z), \tag{8}$$

can be obtained by a variety of methods. See for example, Courant and Hilbert [6] or Copson [7].

For the particularly simple case that the matrix is heated by a unit step change of the inlet gas temperature, and the initial temperature of the matrix is kept constant, one assumes the boundary conditions

$$G(0,\tau) = g(\tau) \equiv 1, \tag{9}$$

$$S(z,0) = s(z) \equiv 0. \tag{10}$$

In this paper, only the case described by equations (9) and (10) will be treated. The exit fluid temperature as determined by Hausen's equations (5) and (6) is then given by

$$G(z,\tau) = e^{-z}\left[1 + z\int_0^\tau \Xi_1(\theta z)\, e^{-\theta}\, d\theta\right]. \tag{11}$$

In equation (11) $\Xi_1(x)$ denotes one function ($k = 1$) of a family of functions $\Xi_k(x)$ of order $k = 0, 1, 2, \ldots$ which are defined by infinite power series:

$$\Xi_k(x) = \sum_{k=0}^{\infty} \frac{x^n}{n!(n+k)!}. \tag{12}$$

They are related to the Bessel and modified Bessel functions of the first kind and order k by

$$\Xi_k(x) = \begin{cases} I_k(2\sqrt{x})/(\sqrt{x})^k, & x \geqslant 0, \\ J_k(2\sqrt{|x|})/(\sqrt{|x|})^k, & x \leqslant 0. \end{cases} \tag{13}$$

For all $\Xi_k(x)$ the following relationship can be verified:

$$\frac{\mathrm{d}\Xi_k(x)}{\mathrm{d}x} = \Xi_{k+1}(x) = \frac{1}{x}[\Xi_{k-1}(x) - k\Xi_k(x)], \qquad x \neq 0. \tag{14}$$

It is also possible to express $G(z,\tau)$ of equation (11) by an infinite series. From

$$\Xi_1(z\theta)\,\mathrm{e}^{-\theta} = \sum_{k=0}^{\infty} \frac{z^k \theta^k}{k!(k+1)!} \times \sum_{l=0}^{\infty} \left(\frac{(-1)^l \theta^l}{l!}\right) = \sum_{l=0}^{\infty} \theta^l \sum_{k=0}^{\infty} \frac{(-1)^{l-k} z^k}{k!(k+1)!(l-k)!} \tag{15}$$

it follows readily that

$$G(z,\tau) = \mathrm{e}^{-z}\left[1 + z\tau \sum_{l=0}^{\infty} \frac{\tau^l}{l+1} \times \left(\sum_{k=0}^{\infty} \frac{(-1)^{l-k} z^k}{k!(k+1)!(l-k)!}\right)\right]$$
$$= \mathrm{e}^{-z}\left\{1 + z\tau\left[1 - \left(1 - \frac{z}{2}\right)\frac{\tau}{2} + \left(1 - z + \frac{z^2}{6}\right)\frac{\tau^2}{6} - \left(1 - \frac{3z}{2} + \frac{z^2}{2} - \frac{z^3}{24}\right)\frac{\tau^3}{24} + \ldots\right]\right\}. \tag{16}$$

It may be remarked here that $G(z,\tau)$ as given by equation (16) is undoubtedly more useful for the purpose of numerical evaluation than Schumann's analytical solution [5].

DETERMINATION OF MAXIMUM SLOPE

The derivative of the exit fluid temperature is

$$\frac{\partial G(z,\tau)}{\partial \tau} = z\,\mathrm{e}^{-(z+\tau)}\,\Xi_1(z\tau) = \mathrm{e}^{-(z+\tau)} \sqrt{\left(\frac{z}{\tau}\right)}\, I_1[2\sqrt{(z\tau)}]. \tag{17}$$

The data in Locke's [4] report are based on an approximation of the expression in (17); in this paper the numerical evaluation of maximum derivatives will be based on exact analysis.

The maximum of $\partial G(z_1\tau)/\partial\tau$ is not, as would be desirable from the point of view of a simple interpretation, a monotonic increasing function of z. For this reason it is useful to change from the independent variable τ to a new independent variable σ, the latter determined by

$$\sigma = \sigma\left(\alpha, \beta, t, \frac{x}{L}\right) = \frac{\tau}{z} = \alpha\frac{L}{x}t - \beta \tag{18}$$

where $\alpha = w_f c_f / W_s c_s$ and $\beta = W_f c_f / W_s c_s$.

The derivative of the exit fluid temperature can now be expressed as

$$\frac{\partial G(z,\tau)}{\partial \sigma} = z^2\,\mathrm{e}^{-z(1+\sigma)}\,\Xi_1(z^2\sigma). \tag{19}$$

The maximum of $\partial G(z,\tau)/\partial\sigma$ which corresponds to the maximum slope of the graph of $G(z,\sigma)$ is, according to Locke, of particular interest:

$$m(z) \equiv \max_{\sigma} \frac{\partial G(z,\sigma)}{\partial \sigma}, \qquad 0 \leqslant \sigma \leqslant 1. \tag{20}$$

In order to determine $m(z)$ consider first the necessary conditions for a relative maximum of $\partial G(z,\sigma)/\partial\sigma$, namely

$$\frac{\partial^2 G(z,\sigma)}{\partial \sigma^2} = z^2\,\mathrm{e}^{-z}\frac{\partial}{\partial\sigma}[\mathrm{e}^{-z\sigma}\,\Xi_1(\sigma z^2)] = z^2\,\mathrm{e}^{-z}\,\mathrm{e}^{-z\sigma}[z^2\,\Xi_2(\sigma z^2) - z\,\Xi_1(\sigma z^2)] = 0. \tag{21}$$

From (21) follows

$$z = \frac{\Xi_1(\sigma z^2)}{\Xi_2(\sigma z^2)}, \qquad \text{all} \qquad z > 0. \qquad (22)$$

Equation (22) defines implicitly a function $\sigma_r(z)$. The latter corresponds to the locus of all points in the (z,σ)-plane on which $\partial G/\partial\sigma$ may have a relative maximum.

For convenience let

$$Q(x) \equiv \frac{\Xi_1(x)}{\Xi_2(x)}; \qquad (23)$$

then

$$\sigma_r(z) = \frac{1}{z^2} Q^{-1}(z), \qquad (24)$$

where $Q^{-1}(.)$ denotes a function inverse to $Q(.)$. $Q(x)$ may be studied in a neighborhood of the origin by the power series

$$Q(x) = \sum_{n=0}^{\infty} c_n x^n, \qquad (25)$$

the coefficients of which are determined by

$$c_n = \left[2 \frac{1}{n!\,(n+1)!} - (1 - \delta_{0,n}) \sum_{i=0}^{n-1} \frac{c_i}{(n-i)!\,(n-i+2)!} \right]. \qquad (26)$$

The series $\sum_{n=0}^{\infty} c_n x^n$ converges for all $|x| <$ 6·5946. The first few terms are

$$Q(x) = 2 + \frac{1}{3}x - \frac{1}{36}x^2 + \frac{1}{270}x^3 - \frac{7}{12960}x^4 + \ldots \qquad (27)$$

It may be noted that $Q(x)$ is also well defined for those values for which the series does not converge, for example by analytic continuation. Thus one finds

$$Q(x) \geqslant 2 \qquad \text{for all} \qquad x \geqslant 0. \qquad (28)$$

Table 1. Maximum slopes

The following table presents (1) the maximum slope m due to a step-function input and (2) the reduced time μ_r (Hausen's variables) which indicates when the maximum slope is reached, as a function of the dimensionless heat-transfer parameter N_{tu}

N_{tu}	m	μ_r
2	0·5413	0·0000
2·25	0·5448	0·1576
2·5	0·5531	0·2714
2·75	0·5641	0·3571
3	0·5766	0·4238
3·25	0·5900	0·4774
3·5	0·6039	0·5214
3·75	0·6180	0·5583
4	0·6321	0·5897
4·25	0·6463	0·6167
4·5	0·6603	0·6403
4·75	0·6743	0·6611
5	0·6880	0·6796
5·5	0·7151	0·7110
6	0·7414	0·7367
6·5	0·7670	0·7581
7	0·7919	0·7763
7·5	0·8161	0·7919
8	0·8397	0·8055
8·5	0·8627	0·8174
9	0·8852	0·8279
9·5	0·9072	0·8373
10	0·9286	0·8457

The above series can be inverted to give

$$x = 3(Q-2) + \frac{3^2}{12}(Q-2)^2 + \frac{3^3}{360}(Q-2)^3 - \frac{3^4}{8640}(Q-2)^4 + \ldots \qquad (29)$$

from (24) and (29) one obtains, in a neighborhood of $z = 2$,

$$\sigma_r(z) = \frac{3(z-2)}{z^2} + \frac{3^2(z-2)^2}{12z^2} + \frac{3^3(z-2)^3}{360z^2} - \frac{3^4(z-2)^4}{8640z^2} + \ldots \qquad z \geqslant 2. \qquad (30)$$

Again, it may be noted that $\sigma_r(z)$ is well-defined even for those values of z for which the series on the right-hand side of equation (30) does not converge. Exact maximum slopes have been obtained by computing $\sigma_r(z)$ and substituting the resulting values $\sigma = \sigma_r$ in formula (19).

Exact values, to four significant figures, of both $m(N_{tu})$ and $\mu(N_{tu}) = \sigma_r|_{x=L}$ are presented in Table 1. They are in best agreement with the values published by Kays and London [2] (Table 3-3, p. 76), but differ from the original data presented by Locke [4] (Table 11, p. 91).

For graphical representation see Figs. 1 and 2.

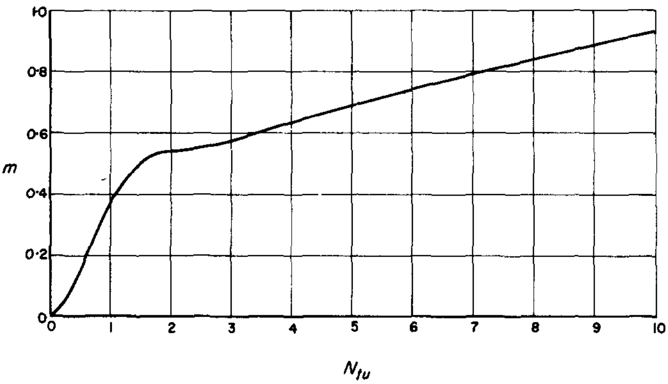

FIG. 1. Maximum slope of fluid temperature transient response: $m(N_{tu})$ vs. N_{tu}.

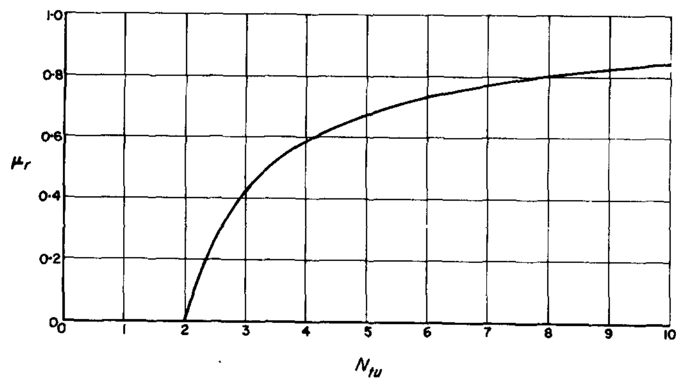

FIG. 2. Reduced time at maximum slope: $\mu_r(N_{tu})$ vs. N_{tu}.

temperature $G(z,\sigma)$ cannot have a point of inflection for $z < 2, \sigma \geqslant 0$; the value $z = 2$ is the greatest lower bound for which inflection points can occur. The latter fact needs to be pointed out explicitly as it is not immediately apparent from a consideration of Locke's [4] approximate analysis, and only implicitly suggested by the published data of Kays and London [2].

In the interval determined by $0 \leqslant z \leqslant 2$ the absolute maximum derivative of $G(z,\sigma)$ is found

SINGULAR BEHAVIOR OF MAXIMUM SLOPE AT $N_{tu} = 2$

Inspection of equation (30) shows (analytically) that the exit fluid temperature has no relative maximum derivative for $z < 2, \sigma \geqslant 0$. Correspondingly, a very smooth graph of the exit fluid

to be

$$m(z) \equiv \max_{\sigma} \frac{\partial G(z,\sigma)}{\partial \sigma} = z^2 \, e^{-z} \, \Xi_1(0) = z^2 \, e^{-z}; \quad (31)$$

the absolute maximum of the derivative of the exit fluid temperature is reached on the line

$\sigma = 0$. The maximum of $\partial G/\partial\sigma$ at the test-section exit is reached at the time $t = \beta/\alpha$, that is in the instant when the temperature front has arrived at the test-section exit.

Theoretically, there is a unique relationship between $\max_\sigma \partial G/\partial\sigma$ and N_{tu} in the interval $0 \leqslant N_{tu} < 2$ provided (i) the inlet fluid temperature change is actually a step function and (ii) the temperature recorders are not lagging. Both of these conditions are hardly ever met in practical experiments. The slope of the exit gas temperature at $t = \beta/\alpha$, in the interval $0 \leqslant N_{tu} < 2$, is the absolute maximum slope. The latter is, for all practical purposes, that is when conditions (i) and (ii) are not strictly satisfied, no reliable measure for the heat-transfer properties of the matrix to be tested.*

The following is of practical interest concerning applications of the maximum slope technique in a small neighborhood of $N_{tu} = 2$ (which corresponds to $z = 2$). The maximum derivative can be represented by

$$m(z) = \max_\sigma \frac{\partial G(z,\sigma)}{\partial\sigma} = \begin{cases} z^2\,\mathrm{e}^{-z}, & 0 \leqslant z \leqslant 2, \\ z^2 \exp\left[-z(1+\sigma_r)\right]\Xi_1(z^2\sigma_r), & z \geqslant 2. \end{cases} \tag{32}$$

The maximum slope $m(z) = z^2\,\mathrm{e}^{-z}$ has, for $z \leqslant 2$, a relative maximum at $z = 2$. For $m(z) = z^2 \exp\left[-z(1+\sigma_r)\right]\Xi_1(z^2\sigma_r)$ one finds the opposite: $m(z)$ has, for $z \geqslant 2$, a relative minimum at $z = 2$. Visual inspection of Fig. 1 suggests at once that the maximum slope, as a function of N_{tu}, has a point of inflection with horizontal tangent at $N_{tu} = 2$. This fact is readily verified by differentiating $m(z)$, as represented by equation (32), with respect to z; for a graphical representation of $\mathrm{d}m(z)/\mathrm{d}z$ see Fig. 3.

For the second derivative of $m(z)$ with respect to z, which is shown in Fig. 4, one finds

$$\lim_{z\to 2^-} \frac{\mathrm{d}^2 m}{\mathrm{d}z^2} = \mathrm{e}^{-2} \neq \lim_{z\to 2^+} \frac{\mathrm{d}^2 m}{\mathrm{d}z^2} = -2\,\mathrm{e}^{-2}. \tag{33}$$

that is, the second derivative of $m(z)$ to the right is positive and different from the second derivative of $m(z)$ to the left which is negative: $\mathrm{d}^2m(z)/\mathrm{d}z^2$ has a finite jump discontinuity at $z = 2$.

Both the horizontal tangent of the maximum derivative at $N_{tu} = 2$ and the finite jump-discontinuity of $\mathrm{d}^2m/\mathrm{d}N_{tu}^2$ is indicative of the singular behavior of the maximum derivative $m(N_{tu})$. However, the singular behavior mentioned above is perhaps most suitably illustrated by the graph shown in Fig. 5 where the quantity

$$K = m\frac{\mathrm{d}(\ln N_{tu})}{\mathrm{d}m} = \frac{m}{N_{tu}}\frac{\mathrm{d}N_{tu}}{\mathrm{d}m} \tag{34}$$

has been plotted versus m. The significance of K will evolve from the following consideration. It is obvious that any method being used to determine N_{tu}-values from fluid temperature response curves is limited by aspects resulting from a complementing error analysis. An analysis of relative errors arising from the application of Locke's [4] maximum slope technique will be given below.

Let m be the experimentally determined maximum slope, Δm the absolute experimental error. The relative experimental error is then $\Delta m/m$. For the absolute error in the N_{tu}-value one has

$$\Delta N_{tu} = \frac{\mathrm{d}N_{tu}}{\mathrm{d}m}\Delta m; \tag{35}$$

for the relative error in the N_{tu}-value one finds

$$\frac{\Delta N_{tu}}{N_{tu}} = \frac{1}{N_{tu}}\frac{\mathrm{d}N_{tu}}{\mathrm{d}m}\Delta m = \frac{m}{N_{tu}}\frac{\mathrm{d}N_{tu}}{\mathrm{d}m}\frac{\Delta m}{m} = K\frac{\Delta m}{m}. \tag{36}$$

* In particular, with respect to condition (i), the author has studied theoretically the effect of a deviation of the inlet fluid temperature change from the step change, on the maximum slope. The result of this investigation, which is to be published elsewhere, shows that even small deviations from the step change may cause the maximum slope to become a multiple-valued non-monotonic function of N_{tu} whose values may differ significantly from the maximum slope due to a step change.

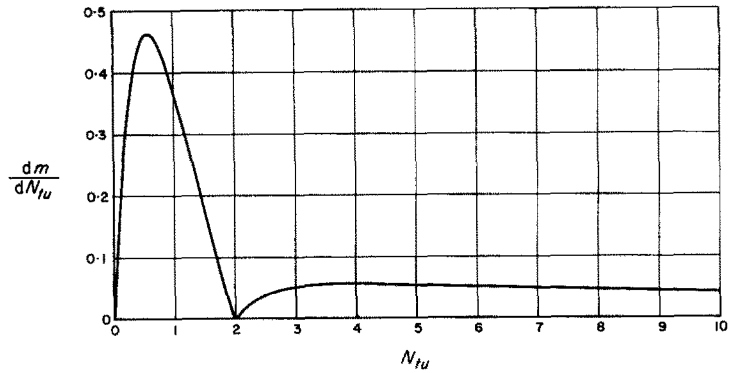

FIG. 3. Derivative of maximum slope with respect to N_{tu}: $\mathrm{d}m/\mathrm{d}N_{tu}$ vs. N_{tu}.

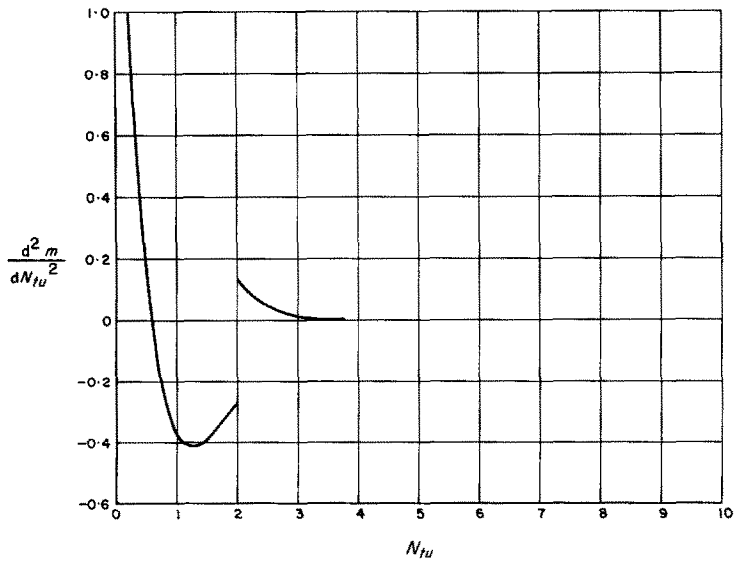

FIG. 4. Second derivative of maximum slope with respect to N_{tu}: $\mathrm{d}^2m/\mathrm{d}N_{tu}^2$ vs. N_{tu}.

Thus K is that factor by which the relative errors in the maximum slope have to be multiplied to obtain relative errors in the N_{tu}-value.

Example: $m = 0{\cdot}545$, $\Delta m = 0{\cdot}02$

$\Delta m/m = 0{\cdot}0367$ (percentage error 3·7 per cent)

$K = 9{\cdot}34$

$\Delta N_{tu}/N_{tu} = 0{\cdot}346$ (percentage error 34·6 per cent)

$N_{tu} = 2{\cdot}257$

$\Delta N_{tu} = 0{\cdot}781$

As shown in Fig. 5, as $m \to 0{\cdot}5413$ ($N_{tu} \to 2$) so $K \to \infty$. Thus the relative errors in N_{tu} are unbounded as N_{tu} approaches the value 2. It may be worthwhile here to point out that as $N_{tu} \to \infty$ so $K \to 2$, that is, the relative error in N_{tu} is, for large values of N_{tu}, about twice the relative error in the maximum slope.

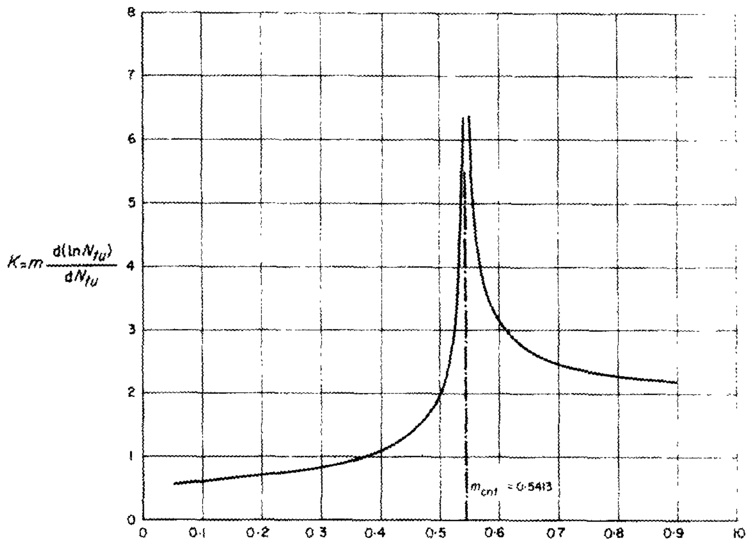

FIG. 5. Amplification of relative errors: $(m/N_{tu})(dN_{tu}/dm)$ vs. m.

The results of the above analysis may be summarized in the following prescription: In order to obtain reliable N_{tu}-values from the application of the maximum slope method

(i) do not use the method in the neighborhood of $m_{crit} = 0{\cdot}5413$ ($N_{tu} = 2$),
(ii) do not use the method for $m < 0{\cdot}5413$ ($N_{tu} < 2$),
(iii) approximate the inlet fluid temperature change as closely as possible by a step change.

One should get satisfactory results using the maximum slope method if the above three conditions are met, in particular whenever the value of K which corresponds to the experiment is close to 2·0 and d^2m/dN_{tu}^2 is negligible.

APPROXIMATION BY ASYMPTOTIC EXPANSIONS

For large values of N_{tu} it is possible to obtain an expression for the maximum derivative by the use of asymptotic expansions. Consider the function (cf. Jahnke and Emde [8])

$$S_p(x) = 1 + \sum_{\nu=1} \{(\nu!)^{-1}(4x)^{-\nu} \prod_{\mu=1}^{\nu} [4p^2 - (2\mu - 1)^2]\}. \quad (37)$$

In terms of this function one has

$$Q(x) = \frac{\Xi_1(x)}{\Xi_2(x)} = (\sqrt{x})\frac{I_1(2\sqrt{x})}{I_2(2\sqrt{x})} \cong (\sqrt{x})\frac{S_1(-4\sqrt{x})}{S_2(-4\sqrt{x})}, \qquad x \gg 0. \quad (38)$$

As a consequence the asymptotic expansion

$$z = Q(z^2\sigma) \cong z\sqrt{\sigma}\frac{\left(1 - \dfrac{3}{16z\sqrt{\sigma}} - \dfrac{15}{512z^2\sigma}\right)}{\left(1 - \dfrac{15}{16z\sqrt{\sigma}} + \dfrac{105}{512z^2\sigma}\right)} \quad (39)$$

defines implicitly $\sigma_r(z)$, again in the sense of an asymptotic expansion. From equation (39) follows, to the same order of approximation

$$\sqrt{\sigma_r} \cong 1 - \frac{3}{4z} - \frac{15}{32z^2} \quad (40)$$

and

$$\sigma_r \cong 1 - \frac{3}{2z} - \frac{3}{8z^2}. \quad (41)$$

Thus one obtains for the maximum derivative, in the sense of an asymptotic expansion,

$$m(z) \cong \frac{1}{2} \sqrt{\left(\frac{z}{\pi}\right)} \exp\left[-z(1 + \sigma_r - 2\sqrt{\sigma_r})\right]$$
$$\left(1 - \frac{3}{16z\sqrt{\sigma_r}} - \frac{15}{512z^2\sigma}\right) \Big/ (\sqrt[4]{\sigma_r})^3. \quad (42)$$

Substitution of equations (40) and (41) leads to the approximation

$$m(z) \cong \frac{\sqrt{z}}{2\sqrt{\pi}} \left(1 + \frac{3}{8z} + \frac{705}{512z^2}\right). \quad (43)$$

For $N_{tu} > 10$ one can thus use the formula

$$m(N_{tu}) \cong 0{\cdot}2823\sqrt{N_{tu}} \left(1 + \frac{0{\cdot}375}{N_{tu}} + \frac{1{\cdot}377}{N_{tu}^2}\right). \quad (44)$$

The functions determined by equations (41) and (44) agree, to sufficient accuracy, with the logarithmic plots given in Kays and London [2] (p. 85, Fig. 3-17). The derivations of this section should be considered as a useful fall-out from the exact analysis.

CONCLUSION

It has been warned against the unrestricted use of Locke's maximum slope technique. Data have been presented which were arrived at by exact analysis. The fact has been pointed out that the maximum slope method has limitations of both theoretical and practical nature for small values of N_{tu}. For $N_{tu} < 2$ no points of inflection exist, for $2 < N_{tu} < 3$ the magnification of relative errors is substantial. For moderate and large values of N_{tu} an approximation may be used which is based on asymptotic expansions.

It is suggested that for small values of N_{tu} a supplementary method for transient heat-transfer test-data evaluation is needed.

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Résumé—L'auteur obtient des expressions analytiques à la fois pour la pente maximale de la température de sortie du fluide (réponse transitoire de température de fluide) due à une variation en échelon de la température du fluide à l'entrée, en se basant sur la solution du modèle mathématique de Hausen, et pour le temps réduit pour lequel la pente maximale est obtenue. Ces expressions sont employées afin de montrer l'existence d'une valeur critique de N_{tu} au-dessous de laquelle les températures de sortie du fluide n'ont pas de point d'inflexion. Des valeurs avec quetre chiffres significatifs sont données à la fois pour le maximum de pente et pour le temps réduit jusqu'à une valeur de N_{tu} égale à 10. Il est souligné que dans le voisinage de $N_{tu} = 2$, même de trés faibles erreurs dans la détermination du maximum de pente peuvent entrainer des erreurs importantes sur les valeurs correspondantes de N_{tu}. Enfin, une expression asymptotique pour le maximum de la pente est présentée, expression utile pour de grandes valeurs de N_{tu}.

Zusammenfassung—Sowohl für die maximale Neigung des Verlaufs der Austrittstemperaturen (instationärer Temperaturverlauf der Flüssigkeit) infolge einer Temperaturänderung beim Eintritt nach einer Schrittfunktion, die auf einer Lösung eines mathematischen Modells nach Hausen beruht, als auch für die reduzierte Zeit nach der die maximale Neigung erreicht wird, sind analytische Ausdrücke abgeleitet. Mit diesen Ausdrücken wird die Existenz eines kritischen N_{tu}-Wertes nachgewiesen bei dessen Unterschreitung die Kurven für die Austrittstemperaturen der Flüssigkeit keine Beugung aufweisen. Bis auf vier Stellen genaue Werte, sowohl für die Maximalneigung als auch für die reduzierte Zeit sind für N_{tu}-

Werte bis zu zehn angegeben. Es wird gezeigt, dass in der Nähe von $N_{tu} = 2$, selbst für sehr kleine Fehler bei der Bestimmung der Maximalneigung, beachtliche Abweichungen in den entsprechenden N_{tu}-Werten auftreten können. Schliesslich wird ein asymtotischer Ausdruck für die Maximalneigung angegeben, der für gross Werte von N_{tu} nützlich ist.

Аннотация—На основе решения математической модели Хаузена выведены аналитические выражения для описания максимального наклона кривой изменения температуры жидкости на выходе в результате ступенчатого изменения температуры жидкости на входе, а также выражение для приведенного времени, которому соответствует максимальный наклон. Эти выражения используются для подтверждения существования критического значения N_{tu}, ниже которого на кривой температуры жидкости на выходе отсутствует точка перегиба. Для максимального наклона и приведенного времени при N_{tu}, равных 10, приводятся значения с точностью до четырех значащих цифр. Отмечается, что в области $N_{tu} = 2$ даже небольшая погрешность в определении максимального наклона дает значительные погрешности в соответствующих значениях N_{tu}. И, наконец, приводится асимптотическое выражение для максимального наклона, используемое при больших значениях N_{tu}.